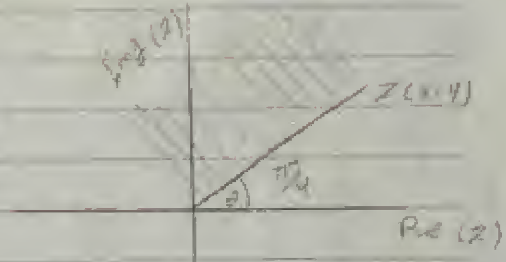
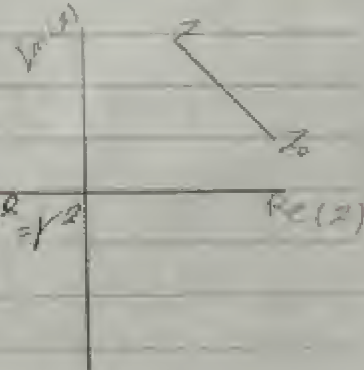
Lect 2

Graphical representation of equations in complex plane

* $\operatorname{Arg} z \geq \pi/4$



→ $|z - z_0| = r$



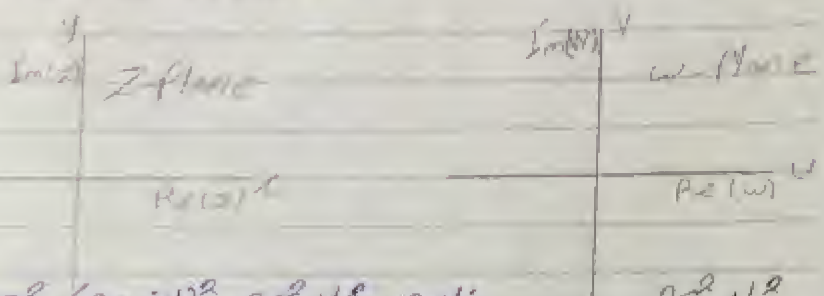
$z = (x, y) \quad z_0 = (x_0, y_0)$

$r = \sqrt{(x-x_0)^2 + (y-y_0)^2} = r \rightarrow (x-x_0)^2 + (y-y_0)^2 = r^2$

Chapter 2

Functions in complex analysis

$w = f(z)$ is a complex function



$w = f(z) = z^2 = (x+iy)^2 = x^2 - y^2 + 2xyi$ $\quad u = x^2 - y^2$

$w = (x^2 - y^2) + i(2xy) = u + iv \rightarrow$ $\quad v = 2xy$

express in u & v for the following

i) $f(z) = e^z$ Sol. $w = f(z) = u + iv = e^z$ but $z = x + iy$

$\therefore w = e^{x+iy} = e^x \cdot e^{iy}$ but $e^{iy} = \cos y + i\sin y$

$w = e^x[\cos y + i\sin y]$ $\therefore w = \underbrace{e^x \cos y}_{u} + i\,\underbrace{e^x \sin y}_{v}$

$\therefore u = e^x \cos y$ $\quad v = e^x \sin y$

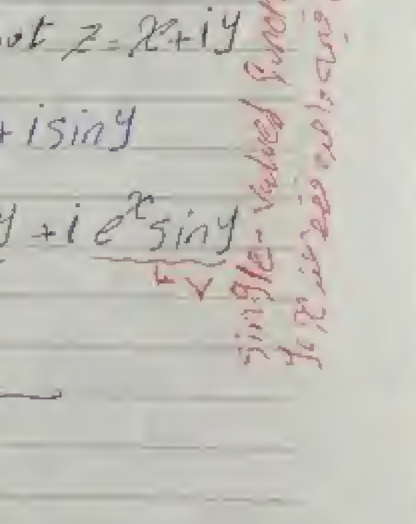
single-valued function

II) ~~w = ln~~ $f(z) = \ln(z)$

$w = \ln(x+iy) = \ln(re^{i\theta})$ & $z = re^{i\theta}$

$w = \ln r + \ln e^{i\theta} \rightarrow i(\theta + 2n\pi)$ $\quad w = \ln r + i(\theta + 2n\pi)$

$u = \ln\sqrt{x^2+y^2}$ $\quad v = (\tan^{-1} y/x + 2n\pi)$



at $n=0 \rightarrow w = \ln\sqrt{x^2+y^2} + i\tan^{-1} y/x$ "Principal value"

$\rightarrow$ multi-valued function

= The limits =

$\lim_{z \to z_0} f(z) = f(z_0) \rightarrow |z - z_0| < \delta \Rightarrow |f(z) - f(z_0)| < \varepsilon$

$f(z)$ has a limit at z_0

ex find $\lim_{z \to 0} (z^2) \rightarrow \lim_{z \to 0} [x^2 - y^2 + i(2xy)]$

* ex $\lim_{z \to 0} \frac{z}{\bar{z}} \rightarrow \lim_{x \to 0} \frac{x + iy}{x - iy}$

$\lim_{z \to 0} \bar{z}/z = \pm 1$

does not exist

= The continuity =

the function $f(z)$ is continuity at $z = z_0$ iff

1) $f(z_0)$ exists 2) $\lim_{z \to z_0} f(z)$ exists

3) $f(z_0) = \lim_{z \to z_0} f(z)$

ex $f(z) = \bar{z} \rightarrow f(z) = x - iy$ at $z = 0$ & $x = 0$ & $y = 0$
$f(0) = 0$

ex 2) $\lim_{z \to 0} \bar{z} = 0$

the Derivation of the complex function

$df/dz = \lim_{\Delta z \to 0} \frac{f(z_0 + \Delta z) - f(z_0)}{\Delta z}$ if the limit is existed then the function is differentiable

and is defined by $\frac{df(z_0)}{dz}$

→ every differentiable function is continous and the inverse isn't true

* $f(z) = \bar{z}$ is continous but not differentiable

because $\lim_{\Delta z \to 0} \frac{(\bar{z_0} + \Delta \bar{z}) - \bar{z_0}}{\Delta z} = \lim_{\Delta z \to 0} \frac{\Delta \bar{z}}{\Delta z}$

→ the limit isn't exised

Analytic function

The function $f(z)$ is called analytic if it's differentiable at the point z_0 and along its neighbour and is called entire if it's differentiable for the whole complex plane

Cauchy-Rieman eq

$$\text{i) } \frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{ii) } \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x}$$

the $f(z) = u + iv$ is analytic and is derivative is

$$\frac{df}{dz} = \frac{\partial u}{\partial x} + i\frac{\partial v}{\partial x} = \frac{\partial v}{\partial y} - i\frac{\partial u}{\partial y}$$

Ex:- Check the analyticy of the following function

1) $f(z) = Re(z^2) \rightarrow f(z) = x^2 - y^2 \quad u = x^2 - y^2 \quad v = 0$

$z^2 = x^2 - y^2 + 2ixy$

not analytic

2) $f(z) = z e^{-z} \rightarrow x e^{-z} + iy e^{-z} = x e^{-x-iy} + iy e^{-x-iy}$

$= x e^{-x}(\cos y - i\sin y) + iy e^{-x}(\cos y - i\sin y)$

$= x e^{-x}\cos y + y e^{-x}\sin y + i[-x e^{-x}\sin y + y e^{-x}\cos y]$

$\therefore u = x e^{-x}\cos y + y e^{-x}\sin y \quad v = -x e^{-x}\sin y + y e^{-x}\cos y$

$u_x = v_y$ & $u_y = -v_x$ the function is analytic

$\frac{df}{dz} = u_x + iv_x$

$= v_y - u_y$

Cauchy-Riemann polar coordinates

$$\text{i) } \frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial v}{\partial \theta} \quad \text{II) } \frac{\partial v}{\partial r} = -\frac{1}{r}\frac{\partial u}{\partial \theta}$$

$\frac{df}{dz} = (u_r + iv_r)(\cos\theta - i\sin\theta)$